Math 275D Lecture 10 Notes

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1 Zeros of Brownian Motion, Time to Exceed a Value, and The Reflection Principle

1.1 Zeros of Brownian motion

Let $T_{t_0} = \inf\{t > t_0 : B(t) = 0\}$ be the first time to hit 0 after time t_0. Based on this, we can define $R_{t_0} = \inf\{t > T_{t_0} : B(t) = 0\}$, the next time to hit 0 after T_0.

Proposition 1.1. *For any t_0, with probability 1, $R_{t_0} = T_{t_0}$.*

Proof. We want to look at $F_{t_0} := \mathbb{1}_{R_{t_0} \neq T_{t_0}}$.

$$\begin{aligned}
\mathbb{E}[F_{t_0}] &= \mathbb{E}[\mathbb{E}[F_0 \circ \theta_{T_{t_0}} \mid \mathcal{F}_{T_{t_0}}]] \\
&= \mathbb{E}[\mathbb{E}[F_0 \mid \mathcal{F}_0]] \\
&= \mathbb{E}[F_0] \\
&= 0.
\end{aligned}$$

□

Define the event $A_q = \{R_q = T_q\}$, where $q \in \mathbb{Q}$. Then $\mathbb{P}(A_q) = 1$, so $\mathbb{P}(\bigcap_{q \in \mathbb{Q}} A_q) = 1$. This means that if $B(t) = 0$ for some t, then there exist either $s_n \nearrow t$ or $s_n \searrow t$ such that $B(s_n) = 0$ (with $n \in \mathbb{Z}$).

Corollary 1.1. *With probability 1, Brownian motion has no isolated zeros.*

1.2 First time to exceed a value

Let $x > 0$, and let $T_x = \min\{t > 0 : B(t) \geq x\}$ be the first time we exceed a. We can think of T_x as a function of x; this function is monotonically increasing. T_x has the property that its values are independent in separate intervals:

Proposition 1.2. *Let $a_1 \leq a_2 \leq b_1 \leq b_2$. Then $(T_{a_2} - T_{a_1}) \perp (T_{b_2} - T_{b_1})$.*

Proof. If $\mathbb{E}[F(A) \mid B]$ is constant for all functions F, then $A \perp B$. So want to show that

$$\mathbb{E}[F(T_{b_2} - T_{b_1}) \mid T_{a_2} - T_{a_1}]]$$

is constant for any function F. This will follow if we can prove that

$$\mathbb{E}[F(T_{b_2} - T_{b_1}) \mid \mathcal{F}_{T_{b_1}}]$$

is constant. We have, by the Strong Markov property, that

$$\mathbb{E}[F(T_{b_2} - T_{b_1}) \mid \mathcal{F}_{T_{b_2}}] = \mathbb{E}_0[F(T_{b_2 - b_1})]. \qquad \square$$

What is the distribution of T_a for some fixed $a > 0$? This is difficult to prove on its own. The correct idea is to study T_x for varying x to find out things about T_a.

Define the **characteristic function** $\varphi_a(\lambda) = \mathbb{E}[e^{-\lambda T_a}]$. Then $\varphi_a(\lambda)\varphi_b(\lambda) = \varphi_{a+b}(\lambda)$ for all λ, as $T_a + T_b \overset{d}{=} T_{a+b}$. The unique solution to this kind of equation is $\varphi_a(\lambda) = e^{-ah_\lambda}$.

How do we find h_λ? Here is a trick: Define $Y_t = e^{\theta B_t - \theta^2 t}$; then Y_t is a Martingale, and $\mathbb{E}[Y_{t\wedge T}] = \mathbb{E}[T_{0\wedge T}] = \mathbb{E}[Y_0] = 1$ for any stopping time T. Then $\mathbb{E}[e^{\theta B_{T_a} - \theta^2 T_a}] = 1$, so we can find h_λ.

1.3 Reflection principle for Brownian motion

We can also ask the following question: What is

$$\mathbb{P}\left(\sup_{t\in[0,1]} B_t \geq a\right)?$$

Surprisingly, we get

$$\mathbb{P}\left(\sup_{t\in[0,1]} B_t \geq a\right) = 2\mathbb{P}(B_1 \geq a).$$

This is called the **reflection principle** for Brownian motion.[1] The idea is that if we hit a, we can reflect the rest of a path above and below the line $y = a$. These paths have the same probability of occurring. So we get

$$\begin{aligned}\mathbb{P}(B_1 \geq a) &= \mathbb{P}\left(B_1 \geq a \mid \sup_{t\in[0,1]} B_t \geq a\right)\mathbb{P}\left(\sup_{t\in[0,1]} B_t \geq a\right) \\ &= \frac{1}{2}\mathbb{P}\left(\sup_{t\in[0,1]} B_t \geq a\right).\end{aligned}$$

[1] Professor Yin has found this principle very useful in his research.